

Matrix superpotentials and superintegrable systems for arbitrary spin ¹

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Abstract

A countable set of quantum superintegrable systems for arbitrary spin is solved explicitly using tools of supersymmetric quantum mechanics. It is shown that these systems (introduced by Pronko [14]) include shape invariant effective potentials classified recently in [16] and [17].

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1 Introduction

Exactly solvable problems of quantum mechanics are very interesting and important. Their solutions can be found in a straightforward way free of uncertainties caused by perturbation methods. The very existence of these solutions is usually connected with symmetries which are very attractive subjects by themselves. Moreover, exact solutions present convenient bases for expansion of solutions of other problems.

Another nice property of some quantum mechanical systems is called superintegrability. The system is called maximally superintegrable if there exist a sufficient number of algebraically independent operators commuting with the Hamiltonian. This number should be equal to $2n - 2$ for the system with n degrees of freedom, and $n - 1$ of these operators should commute amongst each other.

Many of exactly solvable systems are maximally superintegrable, see, e.g., [1], and vice versa. A perfect example of quantum mechanical system which is both maximally superintegrable and exactly solvable is the non-relativistic Hydrogen atom (NRHA). The concept of superintegrability had been extended to the case of quantum systems which include spin [2]. However, the completed relations between the exact solvability and superintegrability are not clear yet, especially for the higher order integrals of motion and for the systems with $n > 2$.

An alternative point of view on the exactly solvable QM systems was created by Gendenshtein [3] who had shown that in many cases such systems admit supersymmetry with shape invariance and can be easily solved using tools of supersymmetric QM. In particular it is true for the NRHA.

A consistent maximally superintegrable and exactly solvable problem was discovered by Pron'ko and Stroganov [4]. The related quantum mechanical system includes a magnetic dipole with spin $\frac{1}{2}$ (neutron) moving in the field of a straight line current. Like the NRHA, this problem admits the hidden symmetry which is more extended than its geometric symmetry [4]. It is supersymmetric also and can be simply solved using the shape invariance of its effective potential [5], [6].

The physical and mathematical aspects of the Pron'ko-Stroganov [4] (PS) model was multiple investigated. Its experimental realizability was discussed in papers [7] and [8], its symmetries and supersymmetries was studied in [9] and [10], enhanced analysis of supersymmetric aspects of this model can be found in [11], etc. Recently the solvable relativistic version of the PS model was found [12] which, however, includes a more complicated external field than in the non-relativistic case.

Whenever there exist a good model for a particle of spin $\frac{1}{2}$, there is a natural desire to generalize it for higher spins. In the case of the PS model the time needed to realize this idea was thirty long years. The main hidden dangers in this generalization were connected with the fact that the direct change of the Pauli interaction term present in the PS model by the analogous term for higher spin leads to the system which is not exactly solvable and its finite trajectories are not closed [13].

In paper [14] exactly solvable generalizations of the PS model to the case of arbitrary

spin have been formulated. The price paid for this progress was the essential complication of the Pauli interaction term present in the PS model. However, there are good physical arguments for such a complication in the case of spin $s > \frac{1}{2}$, in both the non-relativistic [14] and relativistic [15] approaches.

A natural question arises whether the models proposed in [14] can be effectively integrated using the tools of SUSY quantum mechanics, like it has been done [6] for the PS model for neutral particle of spin $\frac{1}{2}$. This question is especially provocative for us since in the recent papers [16] and [17] an effective classification of shape invariant matrix potentials was presented. Thus if the matrix potentials found in [14] are shape invariant they should be nothing but particular cases obtained by this classification.

In the present paper the supersymmetric aspects of the maximally superintegrable systems for arbitrary spin (proposed in [14]) are discussed. It is shown that these systems are shape invariant and can be easily integrated using the tools of SUSY QM. Using the results of paper [16] we construct exact solutions of the mentioned systems directly for arbitrary spin. In addition, a more straightforward and refined formulation of these systems is proposed together with the proper physical interpretation.

2 Hidden symmetry of the PS model

The PS model is based on the following version of the Schrodinger-Pauli Hamiltonian:

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \lambda \frac{S_x y - S_y x}{r^2} \quad (2.1)$$

which is the Hamiltonian of a neutral spinor anomalously interacting with the magnetic field generated by a straight line current directed along the z coordinate axis. Here $p_x = -i\frac{\partial}{\partial x}$, $p_y = -i\frac{\partial}{\partial y}$, $r^2 = x^2 + y^2$; $S_x = \frac{1}{2}\sigma_1$ and $S_y = \frac{1}{2}\sigma_2$ are matrices of spin $\frac{1}{2}$, σ_1 and σ_2 are Pauli matrices, λ is the integrated coupling constant.

The last term in (2.1) is the Pauli interaction term $\lambda \mathbf{S} \cdot \mathbf{H}$ where the magnetic field \mathbf{H} has the following components which we write ignoring the constant multiplier included into the parameter λ :

$$H_x \sim \frac{y}{r^2}, \quad H_y \sim -\frac{x}{r^2}, \quad H_z = 0. \quad (2.2)$$

Hamiltonian (2.1) is invariant w.r.t. rotations around the z -axis since it commutes with the z component of the total angular momentum

$$J_z = x_1 p_2 - x_2 p_1 + S_z \quad (2.3)$$

where $S_z = \frac{1}{2}\sigma_3$. In addition, it admits two more constants of motion [4], namely:

$$A_x = \frac{1}{2}(J_z p_x + p_x J_z) + \frac{m}{r}\mu(\mathbf{n})y, \quad A_y = \frac{1}{2}(J_z p_y + p_y J_z) - \frac{m}{r}\mu(\mathbf{n})x \quad (2.4)$$

where $\mu(\mathbf{n}) = \lambda(S_x n_y - S_y n_x)$, $n_x = \frac{x}{r}$, $n_y = \frac{y}{r}$, and $\mathbf{n} = (n_x, n_y)$. Operators (2.3) and (2.4) commute with \mathcal{H} and satisfy the following commutation relations:

$$[J_z, A_x] = iA_y, \quad [J_z, A_y] = -iA_x, \quad [A_x, A_y] = -J_z \mathcal{H}. \quad (2.5)$$

In other words, operators (2.4) and (2.5) represent algebra $\mathfrak{o}(1,2)$ on the spaces of eigenvectors of \mathcal{H} and cause the degeneration of the Hamiltonian eigenvalues. And this hidden symmetry with respect to algebra $\mathfrak{o}(1,2)$ makes the PS model maximally superintegrable and exactly solvable.

It is interesting to note that up to unitary equivalence operator (3.1) is the only plane Hamiltonian for neutral particle of spin $\frac{1}{2}$ interacting with an external field, which commutes with J_3 and admits a hidden symmetry w.r.t. algebra $\mathfrak{o}(1,2)$. This statement in fact was proven in paper [4] where, however, a more general Hamiltonian has been proposed, namely:

$$\hat{\mathcal{H}} = \frac{p_x^2 + p_y^2}{2m} + a \frac{S_x y - S_y x}{r^2} + b \frac{S_x x + S_y y}{r^2} \quad (2.6)$$

Operator (2.6) does commute with (2.4) and (2.5) and includes two arbitrary parameters a and b . But it is unitary equivalent to (3.1) since

$$\hat{\mathcal{H}} = U \mathcal{H} U^\dagger \quad \text{with} \quad U = \cos \theta + i S_z \sin \theta \quad (2.7)$$

where $S_z = \frac{1}{2} \sigma_3$ and θ is a real parameter such that $a = \lambda \cos \theta$ and $b = \lambda \sin \theta$.

3 SUSY aspects of PS model

Let us apply transformation (2.7). For our purposes it is convenient to set $\theta = \pi/2$, then

$$\hat{\mathcal{H}} = \frac{p_x^2 + p_y^2}{2m} + \lambda \frac{S_x x + S_y y}{r^2}. \quad (3.1)$$

Then we consider the eigenvalue problem for Hamiltonian (3.1):

$$\hat{\mathcal{H}}\psi = \mathcal{E}\psi \quad (3.2)$$

where $\psi = \psi(x, y)$ is a two-component wave function. This function is supposed to be square integrable and vanish at $x = y = 0$.

Introducing the polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta \quad (3.3)$$

and expanding ψ via eigenfunctions of the angular momentum operator J_z :

$$\psi = C_k \psi_\kappa, \quad \psi_\kappa = \frac{1}{\sqrt{r}} \begin{pmatrix} \exp(i(\kappa - \frac{1}{2})\theta) \phi_1 \\ \epsilon \exp(i(k + \frac{1}{2})\theta) \phi_2 \end{pmatrix} \quad (3.4)$$

where C_k are constants, ϕ_1 and ϕ_2 are functions of r and summation is imposed over the repeated indices $k = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2} \dots$, equation (3.2) is reduced to the following system of decoupled equations for ψ_κ :

$$\mathcal{H}_\kappa \psi_\kappa \equiv \left(-\frac{\partial^2}{\partial r^2} + \kappa(\kappa - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{\tilde{\lambda}}{r} \right) \psi_\kappa = \tilde{\mathcal{E}} \psi_\kappa \quad (3.5)$$

where $\tilde{\mathcal{E}} = 2m\mathcal{E}$, $\tilde{\lambda} = 2m\lambda$.

Equation (3.1) and its reduced version (3.5) are invariant with respect to the space reflection $x \rightarrow x, y \rightarrow -y$ provided $\psi(x, y)$ and $\psi_k(r)$ cotransform as

$$\psi(x, y) \rightarrow \Lambda \psi(x, -y), \quad \psi_k(r) \rightarrow \Lambda \psi_{-k}(r) \quad (3.6)$$

where $\Lambda = \sigma_1$.

Transformations (3.6) commute with Hamiltonians \mathcal{H} and \mathcal{H}_κ , anticommute with J_3 and are invertible. Thus we can restrict ourselves to equation (3.5) for positive half integer k . Then solutions for negative k can be obtained using (3.6).

Hamiltonian \mathcal{H}_κ can be factorized as

$$\mathcal{H}_\kappa = a_\kappa^+ a_\kappa + c_\kappa \quad (3.7)$$

where

$$a_\kappa = \frac{\partial}{\partial r} + W_\kappa, \quad a_\kappa^+ = -\frac{\partial}{\partial r} + W_\kappa, \quad c_\kappa = -\frac{\tilde{\lambda}^2}{(2k+1)^2} \quad (3.8)$$

and W is the *matrix superpotential*

$$W_\kappa = \frac{1}{2r} \sigma_3 - \frac{\tilde{\lambda}}{2\kappa+1} \sigma_1 - \frac{(\kappa + \frac{1}{2})}{r}. \quad (3.9)$$

One more important property of \mathcal{H}_κ is its shape invariance, i.e., its superpartner \mathcal{H}_κ^+ is equal to $\mathcal{H}_{\kappa+1}$ up to a constant term:

$$\mathcal{H}_\kappa^+ = a_\kappa^- a_\kappa^+ + c_\kappa = -\frac{\partial^2}{\partial r^2} + (\kappa+1)(\kappa+1-\sigma_3) \frac{1}{r^2} + \sigma_1 \frac{\tilde{\lambda}}{r} = \mathcal{H}_{\kappa+1} + C_\kappa$$

where $C_\kappa = c_\kappa - c_{\kappa+1}$. Thus equation (3.5) can be solved using the standard tools of SSQM. Namely, the ground state vector is defined as a square integrable solution of the first order equation

$$a_\kappa^- \psi_{\kappa,0}(r) \equiv \left(\frac{\partial}{\partial r} + W_\kappa \right) \psi_{\kappa,0}(r) = 0 \quad (3.10)$$

and thus it has the following components

$$\phi_1 = u^{\kappa+1} K_1(u), \quad \phi_2 = u^{\kappa+1} K_0(u) \quad (3.11)$$

where K_0 and K_1 are the modified Bessel functions, $u = \frac{\tilde{\lambda}r}{2\kappa+1}$.

In view of (3.7) function $\psi_{\kappa,0}(r)$ solves also equation (3.5) with $\tilde{\mathcal{E}} = c_\kappa = -\frac{\tilde{\lambda}^2}{(2\kappa+1)^2}$. Solutions which correspond to n^{th} excited state can be represented as

$$\psi_{\kappa,n}(r) = a_\kappa^+ a_{\kappa+1}^+ \cdots a_{\kappa+n-1}^+ \psi_{\kappa+n,0}(x). \quad (3.12)$$

The corresponding eigenvalue $\tilde{\mathcal{E}}_n$ is given by the following formula:

$$\tilde{\mathcal{E}}_n = \sum_{i=0}^{n-1} C_{\kappa+i} = -\frac{\tilde{\lambda}^2}{(2\kappa + 2n + 1)^2}. \quad (3.13)$$

4 Integrable models for neutral vector bosons

The PS model considered in the previous section describes the interaction of a neutral particle of spin $\frac{1}{2}$ with the field of straight line current. How we can generalize Hamiltonian (3.1) (or the initial Hamiltonian (2.1) which is unitary equivalent to (3.1)) to the case of higher spins? The standard idea is simple to change in (3.1) matrices $S_\alpha = \frac{1}{2}\sigma_\alpha$ by matrices of higher spin. However, in this way we obtain a model which is not either exactly solvable [13] or maximally superintegrable.

To obtain a maximally superintegrable analogues of (2.1) for higher spins it is necessary to make a rather non-trivial generalization of the Pauli interaction term in the Hamiltonian (3.1), including multipole magnetic interactions [14].

In this section we present an analogue of the PS model for particle of spin 1. This model is both maximally super integrable and shape invariant. It is based on the following Hamiltonian

$$\mathcal{H}_s = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{r} \mu_s(\mathbf{n}) \quad (4.1)$$

where

$$\mu_s(\mathbf{n}) = \mu_1(\mathbf{n}) = \mu(2(\mathbf{S} \times \mathbf{n})^2 - 1) + \lambda(2(\mathbf{S} \cdot \mathbf{n})^2 - 1). \quad (4.2)$$

Here μ and λ are arbitrary real parameters, $\mathbf{S} \cdot \mathbf{n} = S_1 n_1 + S_2 n_2$ and $\mathbf{S} \times \mathbf{n} = S_1 n_2 - S_2 n_1$, S_1 and S_2 are matrices of spin 1:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.3)$$

Just Hamiltonian (4.1), (4.2) is the absolutely super integrable analogue of (2.1) for spin one. Matrix (4.2) depends on $\mathbf{n} = (\frac{\mathbf{x}}{r}, \frac{\mathbf{y}}{r})$ and satisfies the following conditions:

$$[\mu_s(\mathbf{n}), J_z] = 0, \quad (4.4)$$

$$\mu_s(\mathbf{n})S_z + S_z\mu_s(\mathbf{n}) = 0. \quad (4.5)$$

Thus Hamiltonian (4.1) commutes with operators (2.3) and (2.4) where $\mu(\mathbf{n}) \rightarrow \mu_1(\mathbf{n})$ and S_z is the matrix given in (4.3). So this Hamiltonian admits dynamical symmetry w.r.t. algebra $\mathfrak{o}(1,2)$.

It is possible to show that, up to unitary equivalence, equations (4.1), (4.2) represent the most general form of plane Hamiltonian for neutral particle of spin 1, which admits this symmetry, see section 7. We will see that, in addition, this Hamiltonian is shape invariant. Let us discuss its physical content.

The physical sense of the PS model for spin $\frac{1}{2}$ is absolutely clear. The related Hamiltonian (3.2) includes the Pauli term $\sim \mathbf{S} \cdot \mathbf{H}$ and corresponds to the interaction of a neutral particle (having a non-trivial dipole moment) with the field of the constant and straight line current.

Physical content of superintegrable models for higher spins is much more sophisticated. The interaction term of Hamiltonian (3.2) has nothing to do with the Pauli term and needs another interpretation.

Let us consider in more detail the case $\mu = 0$, $\lambda = \omega > 0$. The corresponding Hamiltonian (4.1) takes the following form:

$$\mathcal{H}_1 = \frac{p_x^2 + p_y^2}{2m} + \omega \left(\frac{2(\mathbf{S} \cdot \mathbf{x})^2}{r^3} - \frac{1}{r} \right). \quad (4.6)$$

and can be represented as

$$\mathcal{H}_1 = \frac{p_{x_1}^2 + p_{x_2}^2}{2m} + \omega Q_{ab} \frac{\partial E_a}{\partial x_b} + \frac{\omega}{3} \text{div} \mathbf{E} \quad (4.7)$$

where

$$Q_{ab} = S_a S_b + S_b S_a - \frac{2}{3} s(s+1) \delta_{ab}$$

is the tensor of quadruple interaction,

$$\mathbf{E} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad (4.8)$$

and the temporary notations $x = x_1, y = x_2$ are used.

In accordance with (4.7) \mathcal{H}_1 can be interpreted as a Hamiltonian of spin-one particle which has neither minimal nor dipole interaction with the external field. However, this particle is supposed to have a quadruple and Darwin interaction, the latest is represented by the last term in (4.7).

A more delicate question is related to the physical realizability of the vector field (4.8) included into the interaction terms. If we suppose that it is the classical Maxwell electric field, the corresponding charge density should be proportional to $\text{div} \mathbf{E} = \frac{1}{r}$. Such

charge density is seemed to be hardly realized experimentally. However, vector (4.8) perfectly solves field equations of generalized Maxwell electrodynamics modified by presence of Chern-Simons term, and also equations of axion electrodynamics with trivial current and charge densities [18].

Let us present one more expression of the interaction term via physical fields. Setting $\lambda = 0, \mu = \omega$ and restricting ourselves to $\mu > 0$ we can rewrite Hamiltonian (4.1) in the following form:

$$\hat{\mathcal{H}}_1 = \frac{p_x^2 + p_y^2}{2m} + \lambda \frac{2(\mathbf{S} \cdot \mathbf{H})^2 - \mathbf{H}^2}{|\mathbf{H}|}, \quad (4.9)$$

where \mathbf{H} is the vector of magnetic field whose components are defined in equation (2.2).

Notice that Hamiltonians (4.6) and (4.9) are unitary equivalent. Namely,

$$\mathcal{H}_s = U \hat{\mathcal{H}}_s U^\dagger, \quad U = \exp\left(\frac{i\pi}{2} S_z\right) \quad (4.10)$$

where $s = 1$.

The last term in (4.9) represents a nonlinear interaction with the well defined external magnetic field (2.2). This field solves the Maxwell equations with a constant straight line current.

Let us remind that a nonlinear generalization of the Pauli interaction is requested also in the relativistic description of spin-one particle interacting with the constant magnetic field [15].

5 Absolutely super-integrable models for spin $\frac{3}{2}$

Let us discuss an exactly solvable model for particle of spin $\frac{3}{2}$. The corresponding Hamiltonian has the generic form (4.1) with

$$\mu_s(\mathbf{n}) = (\nu + \mu S_3^2) (7\mathbf{S} \times \mathbf{n} - 4(\mathbf{S} \times \mathbf{n})^3). \quad (5.1)$$

Here μ and ν are arbitrary parameters, S_x, S_y and S_z are the 4×4 matrices of spin $\frac{3}{2}$ which can be chosen in the following form:

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad S_y = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad (5.2)$$

$$S_z = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Matrix (5.1) satisfies conditions (4.4) and (4.5) which are necessary and sufficient for existence of constants of motion (2.3) and (2.4) [14]. Thus the system whose Hamiltonian is given by equations (4.1) and (5.1) actually is maximally superintegrable.

The interaction term of the considered Hamiltonian can be represented in terms of external fields. In the case $\mu = 0$, $\nu = \frac{\lambda}{3} \neq 0$ we have:

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \omega \left(\mathbf{S} \cdot \mathbf{H} - \frac{4(\mathbf{S} \cdot \mathbf{H})^3}{7\mathbf{H}^2} \right) \quad (5.3)$$

where $\omega = \frac{7\lambda}{3}$, \mathbf{H} is the vector of magnetic field whose components are defined in equation (2.2) and \mathbf{S} is the spin $\frac{3}{2}$ vector with components (5.2).

In addition to the standard Pauli term $\omega \mathbf{S} \cdot \mathbf{H}$ Hamiltonian (5.3) includes the additional interaction $\sim \frac{(\mathbf{S} \cdot \mathbf{H})^3}{\mathbf{H}^2}$ which is non-linear in magnetic field.

Alternatively, for $\mu = -\frac{4}{3}\nu$ we obtain the following representation:

$$\mathcal{H} = \frac{p_{x_1}^2 + p_{x_2}^2}{2m} - \nu Q_{abc} \frac{\partial^2 H_a}{\partial x_b \partial x_c} \quad (5.4)$$

where

$$Q_{abc} = \sum_{P(a,b,c)} \left(S_a S_b S_c - \frac{7}{4} S_a \delta_{bc} \right)$$

is the octuple interaction tensor (the summation is imposed over all possible permutations of indices a, b and c), and $\mathbf{H} = (x_2 \ln r, -x_1 \ln r, 0)$.

The vector field \mathbf{H} perfectly solves equations of the axion electrodynamics [18]. On the other hand, being treated as the classical magnetic field it requires a current whose density grows algorithmically with growing of r .

6 Superintegrable systems for arbitrary spin

In paper [14] the generalization of Hamiltonian (2.1) for arbitrary spin s was proposed, which admit dynamical symmetry w.r.t. algebra $\mathfrak{o}(1,2)$. This Hamiltonian has the generic form (4.1) where $\mu_s(\mathbf{n})$ is a $(2s+1) \times (2s+1)$ dimensional matrix depending on $\mathbf{n} = (\frac{x}{r}, \frac{y}{r})$ and satisfying conditions (4.4), (4.5) with S_z being the z component of spin s vector, i.e., the matrix

$$S_z = \text{diag}(s, s-1, s-2, \dots, -s). \quad (6.1)$$

It can be verified by direct calculations that if conditions (4.4) and (4.5) are satisfied then Hamiltonian (4.1) commutes with operators (2.3) and (2.4) (where S_z is matrix (6.1) and $\mu(\mathbf{n}) \rightarrow \mu_s(\mathbf{n})$), and these operators do satisfy relations (2.5).

We shall refine and extend the results of paper [14]. First, we present a straightforward formulation of Hamiltonians (4.1) for arbitrary spin. Secondly, we prove that the number of

arbitrary parameters present in the models found in [14] can be effectively reduced applying a unitary transformation. At the third place the shape invariance of Hamiltonians (4.1) will be proven and the corresponding eigenvalue problems will be solved algebraically.

Let us find Hamiltonians (4.1) for arbitrary spin. The condition (4.4) is satisfied iff $\mu_s(\mathbf{n})$ is a function of $\mathbf{S} \cdot \mathbf{n} = S_x n_x + S_y n_y$, $\mathbf{S} \times \mathbf{n} = S_x n_y - S_y n_x$ and S_z . At the first step we restrict ourselves to matrices $\mu_s(\mathbf{n})$ which are polynomials in $\mathbf{S} \cdot \mathbf{n}$.

It is convenient to represent matrix $\mu_s(\mathbf{n})$ in the following form:

$$\mu_s(\mathbf{n}) = \sum_{\nu=-s}^s c_\nu \Lambda_\nu \quad (6.2)$$

where c_ν are unknown coefficients and Λ_ν are projectors onto the eigenspaces of matrix $\mathbf{S} \cdot \mathbf{n}$ corresponding to the eigenvalue ν ($\nu, \nu' = s, s-1, \dots, -s$):

$$\Lambda_\nu = \prod_{\nu' \neq \nu} \frac{\mathbf{S} \cdot \mathbf{n} - \nu'}{\nu - \nu'}. \quad (6.3)$$

Matrix (6.2) by construction satisfies condition (4.4). Substituting (6.2) into (4.5) and using the identities

$$S_z \mu_s(\mathbf{n}) + \mu_s(\mathbf{n}) S_z = 2S_z \mu_s(\mathbf{n}) + [\mu_s(\mathbf{n}), S_z]$$

and [19]²

$$[\Lambda_\nu, S_z] = \frac{1}{2} S_z (2\Lambda_\nu - \Lambda_{\nu+1} - \Lambda_{\nu-1}) + \frac{i}{2} (n_x S_y - n_y S_x) (\Lambda_{\nu+1} - \Lambda_{\nu-1}) \quad (6.4)$$

we easily find the following condition for coefficients c_ν (note that projectors Λ_ν are orthogonal and linearly independent):

$$c_\nu = -c_{\nu-1}, \quad \nu = s, s-1, \dots, 1-s. \quad (6.5)$$

In accordance with (6.2) and (6.5) the general expression for $\mu_s(\mathbf{n})$ can be given by the following equation:

$$\mu_s(\mathbf{n}) = \lambda \sum_{\nu} (-1)^{[\nu]} \Lambda_\nu \quad (6.6)$$

where $[\nu]$ is the entire part of ν , $\nu = s, s-1, \dots, -s$.

Equations (4.1), (6.6) present Hamiltonians for arbitrary spin s admitting dynamical symmetry w.r.t. algebra $\mathfrak{o}(1,2)$. Potentials (6.6) are defined up to arbitrary parameter λ which can be associated with the coupling constant.

²Formula (6.4) is a particular case of equation (A.2) in [19] corresponding to $\frac{p_1}{p} = n_x$, $\frac{p_2}{p} = n_y$, $\frac{p_3}{p} = 0$, $S_{23} = S_x$, $S_{31} = S_y$. This formula is also a consequence of equation (7.18) from book [20]

However in accordance with the results of paper [14] the general solution of the determining equations (4.4), (4.5) depends on $2s + 1$ arbitrary real parameters. This statement is in accordance with the fact that there exist exactly $2s + 1$ linearly independent matrices anticommuting with S_z .

But our analysis admits a straightforward and simple extension to the generic case. Indeed, equations (4.4), (4.5) are invariant w.r.t. multiplying $\mu_s(\mathbf{n})$ by arbitrary power of matrix S_z . Thus, starting with (6.6) we can immediately construct $2s + 1$ -parametrical solutions of this equation:

$$\tilde{\mu}_s(\mathbf{n}) = \sum_{\nu \geq 0} (b_\nu \tilde{B}_\nu + i d_\nu \tilde{C}_\nu) \mu_s(\mathbf{n}) \quad (6.7)$$

where $\mu_s(\mathbf{n})$ are matrices (6.6), b_ν and d_ν are arbitrary (real) parameters, \tilde{B}_ν and \tilde{C}_ν are projectors, polynomial in S_z :

$$\tilde{B}_\nu = \prod_{\nu' \neq \nu} \frac{S_z^2 - \nu'^2}{\nu^2 - \nu'^2}, \quad \tilde{C}_\nu = \frac{S_z}{\nu} \prod_{\nu' \neq \nu} \frac{S_z^2 - \nu'^2}{\nu^2 - \nu'^2}, \quad \nu, \nu' = s, s-1, s-2, \dots, \quad \nu, \nu' \geq 0.$$

Notice that C_ν includes only odd powers of S_z and so they anticommute with $\mu_s(\mathbf{n})$. Thus matrix $\tilde{\mu}_s(\mathbf{n})$ is hermitian.

Using the orthogonality relations

$$\tilde{B}_\mu \tilde{B}_\nu = \tilde{C}_\mu \tilde{C}_\nu = \delta_{\mu\nu} \tilde{B}_\nu, \quad \tilde{C}_\mu \tilde{B}_\nu = \delta_{\mu\nu} \tilde{C}_\nu$$

it is easy to prove that all terms in sums (6.7) are linearly independent. Thus matrix $\hat{\mu}_s(\mathbf{n})$ includes exactly $2s + 1$ essential parameters and so it should be equivalent to the analogous matrix found in [14], see equation (29) there. However the number of these parameters can be reduced using the unitary transformation

$$\tilde{\mu}_s(\mathbf{n}) \rightarrow U \tilde{\mu}_s(\mathbf{n}) U^\dagger = \sum_{\nu \geq 0} \lambda_\nu B_\nu \mu_s(\mathbf{n}) \equiv \hat{\mu}_s(\mathbf{n}) \quad (6.8)$$

where $\lambda_\nu = \sqrt{b_\nu^2 + d_\nu^2}$ and $U = \sum_{\nu \geq 0} (\cos \theta_\nu B_\nu + i \sin \theta_\nu C_\nu)$. Taking into account that $\mu_s(\mathbf{n})$ commutes with B_ν and anticommutes with C_ν it is easy to make sure that relation (6.8) is true provided parameters θ_ν satisfy the conditions $b_\nu = \lambda_\nu \cos \theta_\nu$ and $d_\nu = \lambda_\nu \sin \theta_\nu$.

It follows from (6.8) that, up to unitary equivalence, matrix $\hat{\mu}_s(\mathbf{n})$ includes only $s + \frac{1}{2}$ or $s + 1$ arbitrary parameters for half integer or integer spin respectively. A particular case of this statement has been proven in section 2, see equations (2.6) and (2.7).

Thus the Hamiltonian for superintegrable model of arbitrary spin can be represented in the following form:

$$\mathcal{H}_s = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{r} \hat{\mu}_s(\mathbf{n}) \quad (6.9)$$

where $\hat{\mu}_s(\mathbf{n})$ is the matrix defined by equations (6.8) and (6.6). Hamiltonian (6.9) is invariant w.r.t. algebra $\mathfrak{o}(2,1)$ whose basis elements are given by equations (2.3) and (2.4) with $\mu(\mathbf{n}) \rightarrow \hat{\mu}_s(\mathbf{n})$.

Notice that in particular case $s = 1$ Hamiltonian (6.9) is reduced to the operator defined by equations (4.1) and (4.2) where $\mu = \frac{1}{2}(\lambda_0 - \lambda_1)$ and $\lambda = \frac{1}{2}(\lambda_0 + \lambda_1)$. For $s = 3/2$ matrix (6.8) can be transformed to the form (5.1) with $\mu = \frac{1}{6}(\lambda_{\frac{3}{2}} - \lambda_{\frac{1}{2}})$, $\nu = \frac{3}{8}\lambda_{\frac{1}{2}} - \frac{1}{24}\lambda_{\frac{3}{2}}$ by applying transformation (4.10).

7 Dual shape invariance and exact solutions

Let us consider the eigenvalue problem (3.2) for Hamiltonian (4.1), (6.6). Like in section 2 we again introduce polar coordinates (3.3) and expand solutions via eigenvectors of operator J_z (2.3) where S_z is the matrix of spin s defined by equation (6.1). These eigenvectors can be represented as:

$$\psi_\kappa = \frac{1}{\sqrt{r}} \exp(i(\kappa - S_z)\theta) \Phi_\kappa(r) \quad (7.1)$$

where

$$\Phi_\kappa(r) = \text{coulomb}(\phi_s, \phi_{s-1}, \dots, \phi_{-s}). \quad (7.2)$$

Substituting (6.9), (7.1), (7.2) into (3.2) we obtain the following equation for radial functions Φ_κ :

$$\hat{\mathcal{H}}_\kappa \Phi_\kappa \equiv \left(-\frac{\partial^2}{\partial r^2} + V_\kappa \right) \Phi_\kappa = \tilde{\mathcal{E}} \Phi_\kappa \quad (7.3)$$

where $\tilde{\mathcal{E}} = \frac{\mathcal{E}}{2m}$ and

$$V_\kappa = \left((k - S_z)^2 - \frac{1}{4} \right) \frac{1}{r^2} + \tilde{\mu}_s \frac{1}{r}, \quad \tilde{\mu}_s = 2m\mu_s(\mathbf{n})|_{n_y=0}. \quad (7.4)$$

Both the initial equation (3.1) and equation (7.3) are invariant w.r.t. the space reflection transformation (3.6) where $\Lambda = \tilde{\mu}_s(\mathbf{n})/2m\lambda$. Thus it is reasonable to search for solutions of (7.3) for non-negative κ since solutions for negative κ can be obtained using transformation (3.6).

Hamiltonian $\hat{\mathcal{H}}_\kappa$ and matrix S_z^2 commute between themselves and so they have a mutual system of eigenfunctions. Matrix S_z^2 is diagonal and its eigenfunctions ψ_ν corresponding to eigenvalues $\nu^2 = s^2, (s-1)^2, (s-2)^2, \dots$ have two or one non-zero components for $\nu^2 > 0$ and $\nu = 0$ respectively. In the standard representation of spin matrix with diagonal S_z and symmetric S_x ,³, matrix $\tilde{\mu}_s$ is symmetric and antidiagonal, and has the unit non-zero entries.

³See, e.g., [14], eq. (22), or [20] eq.(4.65)

Thus the eigenvalue problem (7.3) can be decoupled to the following equations:

$$\hat{\mathcal{H}}_{\kappa,\nu}\psi_{\kappa,\nu} \equiv \left(-\frac{\partial^2}{\partial r^2} + V_\kappa\right)\psi_{\kappa,\nu} = \tilde{\mathcal{E}}\psi_{\kappa,\nu} \quad (7.5)$$

where $\psi_{\kappa,\nu} = \begin{pmatrix} \phi_\nu \\ \phi_{-\nu} \end{pmatrix}$ are two-component functions, and

$$V_{\kappa,\nu} = \frac{(\kappa - \nu\sigma_3)^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r}\sigma_1, \quad \nu \neq 0. \quad (7.6)$$

For $\nu = 0$ the corresponding reduced potential is one-dimensional:

$$V_{\kappa,0} = \frac{(\kappa)^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r}. \quad (7.7)$$

Potentials (7.6) were discussed in paper [16] where parameter ν was denoted as $\mu + \frac{1}{2}$. These potentials are shape invariant and can be expressed via superpotentials as follows:

$$V_{\kappa,\nu} = W_{\kappa,\nu}^2 - W'_{\kappa,\nu} + c_\kappa \quad (7.8)$$

where

$$W_{\kappa,\nu} = \frac{\nu}{r}\sigma_3 - \frac{\tilde{\lambda}}{2\kappa+1}\sigma_1 - \frac{2\kappa+1}{2r} \quad (7.9)$$

and c_κ is the parameter given in equation (3.8).

Shape invariance of $V_{\kappa,\nu}$ can be easily proven since, in addition to the representation (7.8) the following equation holds true:

$$V_{\kappa,\nu}^+ = W_{\kappa,\nu}^2 + W'_{\kappa,\nu} = V_{\kappa+1,\nu} + C_\kappa \quad (7.10)$$

where $C_\kappa = c_{\kappa+1} - c_\kappa$.

One dimensional potential (7.7) is shape invariant too. It can be represented in the standard form (7.8) with

$$W_{\kappa,0} = -\frac{\tilde{\lambda}}{2\kappa+1} - \frac{2\kappa+1}{2r} \quad (7.11)$$

It is important to note that potential (7.6) is invariant w.r.t. the change $\nu \rightarrow \kappa$, $\kappa \rightarrow \nu$, while superpotential (7.9) *is not invariant w.r.t. this change*, since

$$W_{\nu,\kappa} = \frac{\kappa}{r}\sigma_3 - \frac{\tilde{\lambda}}{2\nu+1}\sigma_1 - \frac{2\nu+1}{2r}. \quad (7.12)$$

Thus, in addition to (7.8), there exist the alternative representations of potential (7.6) via superpotential, i.e., equation (7.8) where $W_{\kappa,\nu}$ is changed by superpotential (7.12):

$$V_{\kappa,\nu} = W_{\nu,\kappa}^2 - W'_{\nu,\kappa} + c_\nu, \quad c_\nu = -\frac{\tilde{\lambda}^2}{(2\nu+1)^2}. \quad (7.13)$$

In other words, potential (7.6) appears to be shape invariant w.r.t. the shifts of two parameters, i.e., κ and ν . This is a particular case of the dual shape invariance phenomena discovered in [16].

Using representations (7.8) and (7.13) we easily find the ground state vectors of Hamiltonian (7.5). Solving equations (3.10) with superpotentials (7.9) and (7.12) we obtain the following components of the ground state vectors $\psi_\nu^0 = \text{column}(\phi_\nu^0, \phi_{-\nu}^0)$, $\nu = s, s-1, s-2, \dots, \nu > 0$:

$$\phi_\nu^0 = d_\nu r^{\kappa+1} K_{\nu+\frac{1}{2}} \left(\frac{\tilde{\lambda} r}{2\kappa+1} \right), \quad \phi_{-\nu}^0 = d_\nu (-1)^{\nu-\frac{1}{2}} r^{\kappa+1} K_{\nu-\frac{1}{2}} \left(\frac{\tilde{\lambda} r}{2\kappa+1} \right), \quad \kappa \geq \nu \quad (7.14)$$

for superpotential (7.9), and

$$\phi_\nu^0 = d_\nu r^{\nu+1} K_{\kappa+\frac{1}{2}} \left(\frac{\tilde{\lambda} r}{2\nu+1} \right), \quad \phi_{-\nu}^0 = d_\nu (-1)^{\kappa-\frac{1}{2}} r^{\nu+1} K_{\kappa-\frac{1}{2}} \left(\frac{\tilde{\lambda} r}{2\nu+1} \right), \quad 0 \leq \kappa < \nu \quad (7.15)$$

for superpotential (7.12), where d_ν are integration constants. Solution for $\nu = 0$, i.e., the component ϕ_0^0 , is given by the following equation:

$$\phi_0^0 = d u^{\kappa+\frac{1}{2}} \exp(-\tilde{\lambda} r), \quad \kappa = 0, 1, 2, \dots \quad (7.16)$$

Functions (7.14) and (7.15) are square integrable for $\tilde{\lambda} > 0$ and arbitrary integer or half integer $\kappa \geq 0$. The same is true for (7.16). However, functions (7.14) for $\kappa \leq \nu$ and functions (7.15) for $\kappa \geq \nu$ do not vanish at $r = 0$. So such values of parameters ν and κ should be excluded, as it is indicated in the r.h.s. of the discussed equations.

Vectors for excited states and the corresponding energy levels again are given by relations (3.12), (3.8) and (3.13) respectively where W_κ should be replaced by superpotential (7.9) or (7.12). In accordance with the analysis presented in [16] all such vectors are square integrable and vanish at $r = 0$.

Thus we find the eigenvectors of Hamiltonian (4.1), (6.6) which are given by equations (7.1), (7.2) and (7.14)–(7.16). The corresponding solutions for negative κ can be easily found using transformation (3.6) where $\Lambda = \tilde{\mu}_s(\mathbf{n})$.

In complete analogy with the above we can solve the eigenvalue problem for the more general Hamiltonian (6.9) including $[s+1]$ arbitrary parameters $\tilde{\lambda}_\nu$. Actually, to this effect it is sufficient to change $\tilde{\lambda} \rightarrow \tilde{\lambda}_\nu$ in all formulae (7.6), (7.9)–(7.15) and (3.13).

8 Discussion

Shape invariant matrix potentials classified in papers [16] and [17] can appear in many realistic integrable models of quantum mechanics. In the present paper we apply the

results of these papers to the interesting class of exactly solvable systems found in [14]. These systems are maximally superintegrable and generalize the PS model to the case of arbitrary spin.

We find it be interesting to study these long-awaited generalized models in more detail. In particular, to search for their solutions, to examine their consistency, and to verify whether the supersymmetry and shape invariance of the spin $\frac{1}{2}$ model are kept in the case of arbitrary spin. In addition, it is important to understand the physical content of the models proposed in [14]. Just the tasks enumerated above are the subjects of the present paper.

First we refine the results of paper [14] taking into account equivalence relations w.r.t. unitary transformations realized by constant matrices. It is shown that up to such equivalence the superintegrable models for arbitrary spin s include $s + 1$ or $s + \frac{1}{2}$ arbitrary parameters for integer or half integer spins correspondingly, while in [14] these models include $2s + 1$ parameter.

The cases $s = 1$ and $s = \frac{3}{2}$ are considered in more detail. We present the corresponding Hamiltonians in the form which is convenient for physical interpretation. This interpretation is proposed using two alternative ways. First, it is possible to treat (4.1) as a Hamiltonian of spin-one particle with zero charge and zero dipole momentum, which has a non-trivial quadruple interaction with the external field. Another possibility is to represent the Hamiltonian in form (4.9) including the interaction term nonlinear in the external magnetic field. Analogous alternatives exist for the models for spin $\frac{3}{2}$ particle considered in section 5 and also for arbitrary spin, i.e., it is possible to interpret the related potentials as results of either multipole or non-linear interaction of a neutral particle with an external field.

It is shown that the considered models are shape invariant. Moreover, their effective potentials appear to be particular cases of the matrix shape invariant potentials classified in [16]. Using this fact it is possible to construct exact solutions of these models in a simple and straightforward way. This program has been realized in section 7 immediately for arbitrary spin.

A specific property of potentials (7.6) is their dual shape invariance, i.e., the existence of two non-equivalent corresponding superpotentials. Exactly this property enables to find good solutions for all combinations of eigenvalues of the spin and orbital momentum operators.

Notice that supersymmetric and superconformal aspects of various planar systems with arbitrary spin were intensively discussed in literature, see, e.g., [21] and [22]. We consider special class of such systems which are chargeless, have non-trivial multipole moments, and can be integrated in closed form using their supersymmetry and shape invariance.

In the present paper the shape invariant matrix potentials classified in [16] and [17] are used to solve explicitly the countable set of maximally superintegrable models. In fact the number of exactly solvable models with matrix superpotentials some of which have the clear physical significance is much more extended. Among them there are also the relativistic models discussed in [12] and many others. We plane to present such models in

the following publications.

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